# Stieltjes Electrostatic Model Interpretation for Bound State Problems

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**Abstract**

In this paper, the duality between the Stieltjes electrostatic interpretation for zeros of orthogonal polynomials and the quantum Hamilton Jacobi formalism is shown. From this duality, the bound state problem mimics as $n$ unit moving imaginary charges $i\hbar$, which are placed in between the two fixed imaginary charges arising due to the classical turning points of the potential. The interaction potential between $n$ unit moving imaginary charges $i\hbar$ is given by logarithm of the wave function. For an exactly solvable potential, this system attains stable equilibrium position at the zeros of the orthogonal polynomials depending upon the interval of the classical turning points.

**keywords :** Orthogonal polynomials, quantum Hamilton Jacobi and zeros of orthogonal polynomials.

## 1 Introduction

Stieltjes [1, 2] considered the following problem with $n$ moving unit charges, interacting through a logarithmic potential, are placed between two fixed charges $p$ and $q$ at $-1$ and

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1 respectively on a real line. He then proved that the system attains a stable equilibrium position at the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Proof is given in Szego's book (section 6.7) [3]. If, the interval is changed on the real line, for the fixed charges, then the the system attains stable equilibrium position at the zeros of the orthogonal polynomial with the respective intervals. For example, in the interval $[0;\infty)$ one gets the Laguerre polynomials $L_n^{(k)}(x)$ and for the the interval $(-\infty;\infty)$ one gets the Hermite polynomials polynomials $H_n(x)$. This model has been extended to the zeros of general orthogonal polynomials in the ref [4].

The Quantum Hamilton Jacobi (QHJ) formalism, was formulated for the bound state problems by Leacock and Padgett [5, 6] and later on was successfully applied to several exactly solvable models (ESM) [7, 8, 9, 10, 11] in one dimension, the quasi - exactly solvable (QES) models [12], the periodic potentials [13] and the PT symmetric potentials [14] in quantum mechanics. In QHJ the central role is played by the quantum momentum function (QMF). This function, in general, contains fixed poles that arises due to the classical turning points of the potential. In general, for most of the potentials in quantum mechanics there will be only two fixed poles, and $n$ moving poles arise due the zeroes of wave function. Thus, one can immediately see the connection between the two scenarios presented above. The fixed poles of the potential are like the the two fixed charges and $n$ moving poles on the real line are like $n$ moving charges. In this letter the duality between the two models in shown in section 2. The brief introduction of these models will follow below.

## 1.1 Electrostatic Model

Stieltjes considered the interaction forces for the $n$ moving unit charges arising from a logarithmic potential which are in between the to fixed charges $p$ and $q$ at $-1$ and 1 respectively on a real line as

$$
\begin{aligned}
L &= -LogD_n(x_1, x_2 ... x_n) + p\sum_{i=1}^{n} Log(\frac{1}{|1-x_i|}) \\
&\quad +q\sum_{i=1}^{n} Log(\frac{1}{|1+x_i|}), \qquad (1)
\end{aligned}
$$

where

$$
-LogD_n(x_1, x_2 ... x_n) = \sum_{1\le i<j\le n}^{n} Log(\frac{1}{|x_i - x_j|}) \qquad (2)
$$

Then, he proved in ref [1, 2] that the expression (1) becomes a minimum when $(x_1, x_2, \cdots, x_n)$ are the zeros of the Jacobi polynomial. For completeness the proof is given as in Szego's book (section 6.7) [3].

The minimum for all the $x_i$ are distinct and different from $\pm 1$ and it occur for $\partial L/\partial x_k = 0 (1 \le k \le n)$ so that one gets the system of equations

$$
\sum_{i=1, i\neq k}^{n} \frac{1}{x_i - x_k} - \frac{p}{x_k - 1} - \frac{q}{x_k + 1} = 0. \qquad (3)
$$

By introducing the polynomial

$$
f_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n), \qquad (4)
$$

then one has

$$
\begin{aligned}
\sum_{i=1, i\neq k}^{n} \frac{1}{x_i - x_k} &= \lim_{x\to x_i} \left[\frac{f_n'(x_k)}{f_n(x_k)} - \frac{1}{x - x_k}\right] \qquad (5) \\
&= \lim_{x\to x_i} \left[\frac{(x - x_k)f_n'(x_k) - f_n(x_k)}{(x - x_k)f_n(x_k)}\right], \qquad (6)
\end{aligned}
$$

and using the L'Hospital's rule

$$
2\sum_{i=1, i\neq k}^{n} \frac{1}{x_i - x_k} = \frac{f_n''(x_k)}{f_n'(x_k)}, \qquad (7)
$$

then this becomes

$$
\frac{1}{2}\frac{f_n''(x_k)}{f_n'(x_k)} + \frac{p}{x_k - 1} + \frac{p}{x_k + 1} = 0. \qquad (8)
$$

This means that the polynomial

$$(1-x^2)f_n''(x)+2[q-p-(p+q)x]f_n'(x)=0 \tag{9}$$

vanishes at the points $x_k$ and since this polynomial is of degree $n$ it must be a multiple of $f_n(x)$. The factor is easily obtained by equating the coefficient of $x_n$ and we have

$$(1-x^2)f_n''(x)+2[q-p-(p+q)x]f_n'(x)=n[n+2(p+q)-1]f_n(x) \tag{10}$$

which is the differential equation for the Jacobi polynomial $P_n^{(2p-1,2q-1)}(x)$. The zeros of the Laguerre and the Hermite polynomials admit the same interpretation.

### 1.2 Quantum Hamilton Jacobi

In this section, a brief review of Quantum Hamilton Jacobi formalism is presented below. For details see the references [11, 9]. The Schrödinger equation is given by,

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z)+V(x,y,z)\psi(x,y,z)=E\psi(x,y,z). \tag{11}$$

One defines a function $S$ analogous to the classical characteristic function by the relation

$$\psi(x,y,z)=\exp\left(\frac{iS}{\hbar}\right) \tag{12}$$

which, when substituted in (11), gives

$$(\vec{\nabla}S)^2-i\hbar\vec{\nabla}.(\vec{\nabla}S)=2m(E-V(x,y,z)). \tag{13}$$

the quantum momentum function $p$ is defined in terms of the function $S$ as

$$\vec{p}=\vec{\nabla}S. \tag{14}$$

Substituting (14) in (13) gives the QHJ equation for $\vec{p}$ as

$$(\vec{p})^2-i\hbar\vec{\nabla}.\vec{p}=2m(E-V(x,y,z)) \tag{15}$$

and from (11) and (14), one can see that $\vec{p}$ is the the logarithmic derivative of $\psi(x, y, z)$ *i. e.*

$$\vec{p} = -i\hbar\vec{\nabla} ln\psi(x, y, z) \tag{16}$$

The above discussion of the QHJ formalism is done in three dimensions the same equation in one dimension takes the following form

$$p^2 - i\hbar\frac{dp}{dx} = 2m(E - V(x)), \tag{17}$$

which is also known as the Riccati equation. In one dimension the eq (16) take the form

$$\vec{p} = -i\hbar\frac{d}{dx} ln\psi(x). \tag{18}$$

It is shown by Leacock and Padgett [5, 6] that the action angle variable gives rise to exact quantization condition

$$J(E) \equiv \frac{1}{2\pi}\oint_C p dx = n\hbar. \tag{19}$$

## 2 Duality

By considering the form of the wave function, in the equation (18), to be $\psi = \prod_{i=1}^{N}(x - x_i)$. Then, in the quantum momentum function this corresponds to $n$ zeros on the real line. These zeros are also called the moving poles in the language of QHJ. Choosing, an exactly solvable potential $V(x)$, with two fixed poles as the classical turning points, substituting in equation (17). Then, for bound states the following feature always arises in QHJ that the $n$ moving poles lie in between the two fixed poles and the solutions are the orthogonal polynomials for the exactly solvable potential $V(x)$. The examples are the Harmonic oscillator, the Coulomb potential , the Scarf potential etc [9, 11].

Thus, the connection between the QHJ and the Stieltjes electro static model can be seen. The fixed poles of the potential are like the two fixed charges and the $n$ moving poles of the real line are like $n$ moving charges. In the electrostatic model the moving charges interact

with the logarithmic potential and in the QHJ the logarithmic potential arises from the wave function. As the quantum momentum function is log derivative of the wave function.

Starting with the QMF the duality between the two models is established The fact that only the residues of the QMF are required for finding the eigenvalues is studied in ref [7, 8]. The formalism for effectively obtaining both the eigenfunctions and the eigenvalues from the singularity structure of the quantum momentum function is given in ref [9]. The quantum momentum function assume that there are no other singular points of $p$ in the complex plane. Then the quantum momentum function is given by [7, 8, 9, 10, 11, 12]

$$p = \sum_{k=1}^{n} \frac{-i}{x - x_k} + Q(x), \tag{20}$$

here the moving poles are simple poles with residue $-i\hbar$ (we take here $\hbar = m = 1$) [9, 11] and $Q(x)$ is the residues of fixed poles arising due to the exactly solvable potentials. This equation resembles the equation (3) except that it is the minimum of the potential. Thus, the quantum momentum function can interpret as system of equations arising for the logarithmic derivative of wave function and fixed poles arising from the classical turning points. By asking the following question, when does this system come to stable equilibrium ? From the above discussion it is clear that answer can be obtained using Stieltjes Electrostatic model. It can be shown that the same wave function can be obtained from both the models. Thus, their exist a duality between the Stieltjes electrostatic interpretation for zeros of orthogonal polynomials and the quantum Hamilton Jacobi formalism.

By rewriting the equation (20) and solving it as a system of equations

$$p = \sum_{k=1}^{n} \frac{1}{x - x_k} + iQ(x) = 0, \tag{21}$$

and applying the L'Hospital's rule

$$2 \sum_{i=1, i\neq k}^{n} \frac{1}{x_i - x_k} = \frac{f_n''(x_k)}{f_n'(x_k)}, \tag{22}$$

then

$$\frac{1}{2}\frac{f_n''(x_k)}{f_n'(x_k)} + iQ(x) = 0. \tag{23}$$

By demanding the points $x_k$ to vanish, the solution to equation (23) will be in terms of the orthogonal polynomial for an exactly solvable potentials. The interval is fixed by the fixed poles of the potential. It is well known that the classical orthogonal polynomials arise as solutions to the bound states problems. Thus, the classical orthogonal polynomials are classified into three different categories depending upon the range of the polynomials. The polynomials in the intervals $(-\infty;\infty)$ are the Hermite polynomials, in the intervals $[0;\infty)$ are the Laguerre polynomials and in the intervals $[-1;1]$ are the Jacobi polynomials. Their singularity structure is as follows $Q(x) = x$ , $Q(x) = \frac{b}{x} + C$, and $Q(x) = -\frac{a}{x-1} - \frac{b}{x+1}$ for the Hermite, the Laguerre and the Jacobi polynomials respectively. Hence, the differential equation can be obtained by examining at the singularity structure of the quantum momentum function. This can be seen by rewriting the eq (23) as

$$f_n''(x) + 2iQ(x)f_n'(x) = 0. \tag{24}$$

The function $Q(x)$ which has the information of fixed pole singularity structure appears as the coefficient of $f_n'(x_k)$. By examining the differential equations of the Hermite, the Laguerre and the Jacobi polynomials the coefficients of $Q(x)$ are fixed.

Let $f(x) = L_\lambda^m(x)$ satisfy the Laguerre differential equation

$$x\frac{d^2}{dx^2}f(x) + (m+1-x)\frac{d}{dx}f(x) + \lambda f(x) = 0, \tag{25}$$

where $\lambda$ is an integer. By examining the first two terms of the differential equations (24) and (25) one gets

$$2iQ(x) = \frac{(m+1)}{x} - 1 \tag{26}$$

the singularity structure for the Laguerre is

$$Q(x) = \frac{b}{x} + C \tag{27}$$

thus one gets $b = -i(m+1)$ and $C = i$. Similarly for the Jacobi differential equation

$$(1-x^2)f_n''(x_k) + 2[p-q-(p+q)x]f_n'(x_k) + n[n+2(p+q)-1]f_n(x) = 0 \tag{28}$$

again comparing the first two terms

$$2iQ(x) = -\frac{p}{x_k - 1} - \frac{q}{x_k + 1} \tag{29}$$

the singularity structure for the Jacobi is

$$2iQ(x) = -\frac{a}{x - 1} - \frac{b}{x + 1} \tag{30}$$

thus one has $p = -ia$ and $q = -ib$. Similar analysis can be done for the Hermite polynomials.

The values of $m$, $p$ and $q$ has to be determined as these are not points like in the electrostatic model. The method adopted by QHJ, search for the polynomial solutions leads to quantization, are used to calculate these values. By writing the quantum momentum function as

$$p = \sum_{k=1}^{n} i\frac{f'(x)}{f(x)} + Q(x) \tag{31}$$

and substituting in (17) then one gets

$$f_n''(x_k) + 2iQ(x)f_n'(x_k) + [Q^2(x) - iQ'(x) - E + V(x)]f(x) = 0. \tag{32}$$

by demanding $[Q^2(x)-iQ'(x)-E+V(x)]$ to be constant, the values of the residues appearing for fixed poles are obtained, in the process the system is quantized. Thus by comparing the equation (24) and (32) it can seen that the singularity structure of $iQ(x)$ determines the differential equation. Therefore, the same wave function is obtained from both the methods.

# 3 Discussion

From the previous discussion, it is clear that the two models are dual to each other. Therefore, From this duality, the bound state problem mimics as $n$ unit moving imaginary charges $i\hbar$, which are placed in between the two fixed imaginary charges arising due to the classical turning points of the potential. The interaction potential between $n$ unit moving imaginary charges $i\hbar$ is given by logarithm of the wave function. For an exactly solvable potential,

this system attains stable equilibrium position at the zeros of the orthogonal polynomials depending upon the interval of the classical turning points. Once charges arise in any model they satisfy the continuity equation of the form

$$\frac{\partial}{\partial t}\rho + \nabla \cdot J = 0. \tag{33}$$

Since, the equation (24) and (32) are nothing but the different form of the Schroedinger equation. Therefore their exist a continuity equation of this form for these imaginary with $\rho = \int_V \psi^* \psi dV$ is probability density function and $J = \frac{\hbar}{i}[\psi^*(\nabla\psi) - \psi(\nabla\psi^*)]$ is probability current density function. Hence, the conservation of probability leading to conservation of imaginary charge and probability current leads to current density for imaginary charge. In this model $\rho$ is the amount of imaginary charge and $J$ is the current density for imaginary charge. Thus, this model is consistent with quantum mechanics.

# 4 Conclusion

In this paper, the two different models, one the Stieltjes electrostatic model and the other one Quantum Hamilton Jacobi formalism are examined. It is shown that the two models are dual to each other except that one is a classical model and another is a quantum model. One new feature comes out of this study is that the wave function can be obtained from the quantum momentum function itself, one need not solve the quantum Hamilton Jacobi equation. From Stieltjes electrostatic model gives nice insights to the methodology of quantum Hamilton Jacobi formalism. It is interesting to note that the Stieltjes electrostatic model existed almost 30 years before quantum mechanics came into existence.

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